SOLUTIONS OF PROBLEMS IN NO. 9.

Solutions of problems in No. 9 have been received as follows:

From A. L. Baker, 38 & 40; Geo. M. Day, 38 & 39; Prof. A. B. Evans, 37, 38, 39 & 40; Henry Gunder, 38 & 39; Henry Heaton, 38, 39 & 40; Artemas Martin, 37 & 38; Walter Siverly, 38 & 39; E. B. Seitz, 38 & 40. S. W. Salmon and R. M. DeFrance each sent an elegant solution to Werner Stille's quest in Curves of Pursuit.

37.—" Divide unity into three parts such that if each part be increased by unity the sums shall be three rational cubes."

SOLUTION BY ARTEMAS MARTIN, ERIE, PA.

Let $a^3x^3 - 1$, $b^3x^3 - 1$ and $c^3x^3 - 1$ be the parts required.

Then $a^3x^3 - 1 + b^3x^3 - 1 + c^3x^3 - 1 = 1$,

or $$(a^3 + b^3 + c^3)x^3 = 4 = A \ldots\ldots (1).$$

$$\therefore \quad x^3 = \frac{A}{a^3 + b^3 + c^3} \ldots\ldots (2).$$

Let $a^3 + b^3 = t$, then $x^3 = \dfrac{A}{t + c^3} \ldots\ldots (3).$

Let $p - s = e$. Then, putting $r^3A = t - s^3$,

$$x^3 = \frac{A}{p^3 - 3p^2s + 3ps^2 + r^3A} = \frac{1}{\frac{p^3}{A} - \frac{3p^2s}{A} + \frac{3ps^2}{A} + r^3}.$$

Assume $$\frac{p^3}{A} - \frac{3p^2s}{A} + \frac{3ps^2}{A} + r^2 = \left(r + \frac{ps^2}{r^2A}\right)^3;$$

by reduction, $$p = \frac{3r^3As}{r^3A - s^3}.$$

$$\therefore \; c = s\left(\frac{2r^3A + s^3}{r^3A - s^3}\right), \; x = \frac{r^2A}{r^3A + ps^2} = \frac{1}{r}\left(\frac{r^3A - s^3}{r^3A + 2s^3}\right),$$

and $\dfrac{a^3}{r^3}\left(\dfrac{r^3A - s^3}{r^3A + 2s^3}\right)^3 - 1$, $\dfrac{b^3}{r^3}\left(\dfrac{r^3A - s_3}{r^3A + 2s^3}\right)^3 - 1$, $\dfrac{s^3}{r^3}\left(\dfrac{2r^3A + s^3}{r^3A + 2s^3}\right)^3 - 1$,

are the parts required.

We must now solve the equation $a^3 + b^3 = t$. $y^3 + z^3 = (y + z)(y^2 - yz + z^2)$.

Suppose $y^2 - yz + z^2 = m^3$, and y and z prime to each other; then $\left(\frac{y}{m}\right)^3 + \left(\frac{z}{m}\right)^3 = y + z$, and in general

$$\left(\frac{y - (n+1)z}{m}\right)^3 + \left(\frac{(n+1)y - nz}{m}\right)^3 = (n^2 + n + 1)[(n+2)y - (2n+1)z].$$

When $n = 0$, $$\left(\frac{y - z}{m}\right)^3 + \left(\frac{y}{m}\right)^3 = 2y - z \ldots\ldots\ldots\ldots (4).$$

Hence we may take $a = \frac{y - z}{m}$, $b = \frac{y}{m}$, and $t\ 2y - z$.

Put $y = v + w$, $z = v - w$; then $y^2 - yz + z^2 = v^2 + 3w^2 = m^3$, which is satisfied by

$$v = p(p + 3q)(p - 3q),\ w = 3q(p + q)(p - q),\ m = p^2 + 3q^2;$$
$$\therefore y = p(p + 3q)(p - 3q) + 3q(p + q)(p - q),$$
$$z = p(p + 3q)(p - 3q) - 3q(p + q)(p - q).$$

Take $p = 11$, $q = 2$; then $y = 1637$, $z = 233$, $m = 133$ and $t = 3041$; $a = \frac{1404}{133}$, $b = \frac{1637}{133}$.

But we must have $r^3A + s^3 = 4r^3 + s^3 = t = 3041$.

Let $r = 9$, then $2916 + s^3 = 3041$, $s^3 = 125$, and $s = 5$. $\therefore x = \frac{2791}{28494}$, and the parts are

$$\left(\frac{3918564}{3789702}\right)^3 - 1 = \frac{5743015291812773736}{54427098504275016408},$$

$$\left(\frac{3961405}{3789702}\right)^3 - 1 = \frac{7738158893915488717}{54427098504275016408},$$

$$\left(\frac{4568867}{3789702}\right)^3 - 1 = \frac{40945924318546753955}{54427098504275016408}.$$

See *Mathematical Miscellany*, pp. 118–123, whence the substance of this solution is taken.

38.—"Prove that $\frac{1}{\underline{|2}} - \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} - \ldots\ldots \frac{(-1)^x}{\underline{|x}} = \frac{1}{e}$; where e is the base of Napierian logarithms."

SOLUTION BY E. B. SEITZ, GREENTILLE, OHIO.

By a well known theorem,

$$e^x = 1 + \frac{x}{1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \&c. \quad \text{Making } x = -1, \text{ we have}$$

$$\frac{1}{\lfloor 2} - \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} - \ldots\ldots\ldots\ldots \frac{(-1)^x}{\lfloor x} = e^{-1} = \frac{1}{e}. \quad \text{Q.E.D.}$$

39.—"Two weights are connected by a fine string which passes over a pulley; if the weights be 50 and 72 lbs., determine what stationary weight the string must be able to support, that it may just escape breaking during the motion."

SOLUTION BY HENRY GUNDER, GREENVILLE, OHIO.

As illustrated by Attwood's Machine, we have a moving force $= 22g$, and a mass to be moved $= 122$; hence the velocity acquired in one second is $\frac{22}{122}g = \frac{11}{61}g$. The 72 pound weight will have acquired this velocity in one second, but moving freely under the influence of gravity its velocity would have been g. $\therefore$ there must have been an opposing force of $\frac{50}{61}g$, or $\frac{50}{61}$ of 72, on the string $= 59\frac{1}{61}$ lbs.

40.—(See ANALYST No. 9.)

SOLUTION BY PROFESSOR ASHER B. EVANS, LOCKPORT, N. Y.

We will take for the axis of y that side of the canal along which A walks and for the axis of x a line perpendicular thereto at an arbitrary point, the canal being on A's right hand side and in the direction of x positive.

Let the time t be computed from an arbitrary instant, and let x' and y' be the coordinates of B when A is on the axis of y at a distance from the origin expressed by $b + nt$, b being an arbitrary constant depending on the situation of the origin and the instant at which the time t is supposed to commence. Then the right line joining B and A will have for its equation $x - x' = \dfrac{x'}{y' - b - nt}(y - y' \ldots\ldots.(1)$; from which it follows that this line will make with the axes of x and y angles whose

cosines are $\frac{x'}{\sqrt{x'^2+(y'-b-nt)^2}}$ and $\frac{y'-b-nt}{\sqrt{x'^2+(y'-b-nt)^2}}$. Two constant forces are acting upon B, the force r due to his rowing and acting in the direction BA, and the force m due to the current and acting in the direction opposite to y positive. The components of the force r in the direction of x and y are $\frac{rx'}{\sqrt{x'^2+(y'-b-nt)^2}}$ and $\frac{r(y'-b-nt)}{\sqrt{x'^2+(y'-b-nt)^2}}$. Considering that the former of these components constantly tends to diminish the coordinate x', and that the latter tends to diminish y' when $y'>(b+nt)$ and to increase it when $y'<(b+nt)$, for the motion of B we shall have, from well known principles of mechanics, on omitting the accents as henceforth useless,

$$\frac{dx}{dt}=-\frac{rx}{\sqrt{x^2+(y-b-nt)^2}} \text{ and } \frac{dy}{dt}=-\frac{r(y-b-nt)}{\sqrt{x^2+(y-b-nt)^2}}-m.(2).$$

To integrate equation (2) put $y-b-nt=x\tan\phi$..............(3);

$$\text{then } \frac{dx}{dt}=-r\cos\phi \text{ and } \frac{dy}{dt}=-r\sin\phi-m\text{............}(4).$$

Differentiating (3) and dividing by dt, we obtain

$$\frac{dy}{dt}-n=\frac{dx}{dt}(\tan\phi)+\frac{x}{dt}d(\tan\phi)\text{............}(5).$$

Substituting in (5) the values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ given by (4),

$$m+n=-\frac{x}{dt}d(\tan\phi)\text{.........................}(6).$$

$$\text{But } \frac{dx}{dt}=-r\cos\phi=-\frac{r}{\sqrt{1+\tan^2\phi}}\text{.........}(7),$$

and from (6) and (7) on putting $\frac{m+n}{r}=p$ we find

$$p\frac{dx}{x}=\frac{d(\tan\phi)}{\sqrt{1+\tan^2\phi}}\text{.........................}(8).$$

Integrating (8) $\log\left(\frac{x}{d}\right)^p=\log\{\tan\phi+\sqrt{1+\tan^2\phi}\}$, or better

$$\left(\frac{x}{c'}\right)^p = \tan\phi + \sqrt{1 + \tan^2\phi} \;\ldots\ldots\ldots\ldots\ldots (9),$$

where c' is an arbitrary constant. To determine this constant, we observe that at a certain time the line joining B and A is perpendicular to the side of the canal, or to the axis of y. Taking the origin, which we have not hitherto completely fixed, at the point where this perpendicular meets the axis of y, denoting the width of the canal by c, and taking the instant when A is at the origin of coordinates for the origin of t, we have when $t = 0$, $b = 0$, $y = 0$ and $x = c$; whence from equation (3) $\tan\phi = 0$, and then from equation (9) $\left(\frac{x}{c'}\right)^p = \left(\frac{c}{c'}\right)^p = 1$ and $c' = c$ the width of the canal.

Equation (9) now gives $\tan\phi = \frac{1}{2}\left\{\left(\frac{x}{c}\right)^p - \left(\frac{x}{c}\right)^{-p}\right\}$; and therefore

$$\sin\phi = \frac{\left(\frac{x}{c}\right)^p - \left(\frac{x}{c}\right)^{-p}}{\left(\frac{x}{c}\right)^p + \left(\frac{x}{c}\right)^{-p}},\quad \cos\phi = \frac{2}{\left(\frac{x}{c}\right)^p + \left(\frac{x}{c}\right)^{-p}} \;\ldots\ldots (10).$$

Eliminating dt from equation (4)

$$\frac{dy}{dx} = \frac{r\sin\phi + m}{r\cos\phi} = \left(\frac{r+m}{2r}\right)\left(\frac{x}{c}\right)^p - \left(\frac{r-m}{2r}\right)\left(\frac{x}{c}\right)^{-p} \ldots\ldots (11).$$

Integrating equation (11), remembering that $x = c$ when $y = 0$, we have

$$\frac{2ry}{c} = \frac{r+m}{p+1}\left\{\left(\frac{x}{c}\right)^{p+1} - 1\right\} + \frac{r-m}{p-1}\left\{\left(\frac{x}{c}\right)^{1-p} - 1\right\} \ldots (12).$$

When $y = a$, $x = 0$ and equation (12) gives $a = \frac{cnr}{[r^2 - (m+n)^2]}$; and therefore $c = \frac{a}{nr}[r^2 - (m + n)^2] =$ the width of the canal.

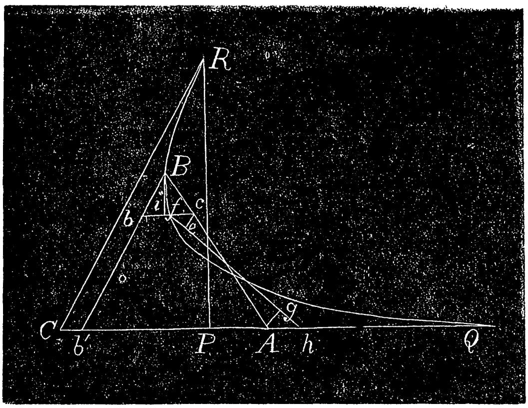
NOTE.—The foregoing general solution includes the problem proposed by Werner Stille on p. 145 of THE ANALYST. For putting $m = 0$, $r = n^2$ and $c =$ the line AB, we have from equation (12)

$$y = \frac{c}{2}\left[\frac{n}{n+1}\left\{\left(\frac{x}{c}\right)^{\frac{n+1}{n}} - 1\right\} - \frac{n}{n-1}\left\{\left(\frac{x}{e}\right)^{\frac{n-1}{n}} - 1\right\}\right]$$

for the equation of the curve pursued by the dog.

[Messrs. A. L. Baker, of Lafayette College, Easton, Pa., and H. Heaton, of Des Moines Public Schools, solved No. 40 by determining the Eq. to the curve which represents the *trace* of the boat in the water, in which case the question may be solved exactly as Mr. Stille's question, by substituting $m + n$ for the velocity of the man ($= 1$ in Mr. Stille's quest.) and r for n, the velocity of the dog.

The foregoing elegant solution of No. 40 is all that could be desired as a solution to questions of this kind. We add, however, the following solution based on geometnical considerations, hoping that it may assist the student in comprehending the solution of these questions:



Put the required distance PR $= p$, and produce QP to C, making PC : PQ :: $m : n$, then is PC $= ma \div n = a'$.

Let A and B represent contemporaneous positions of the two men at the end of any time t. Join RC, and draw Bb parallel to RC. Take a point c on the line AB infinitely near B, and draw $b'c$ parallel to CQ. Put C$b = x$ and AB $= y$.

As the distance traversed by the boat from B towards A, if not influenced by the current, would be proportional to the time, we shall have B$c = rdt$; but during the time dt the current will carry the boat in the direction of the line cb' a dittance $cf = mdt$, the resultant of the two forces being the curve Bf; $\therefore b'f = dx$. Draw Bi perpendicular to $b'c$, then is $b'i = mdt$, $\therefore ci = b'f = dx$, and $fi = dx - mdt$.

If while B describes the curve Bf, A advances to h, we shall have Ah $= ndt$. Join fh and draw Ag and ce each perpendicular to fh. Then by similar triangles we have, Bc : ci :: cf : ef, or $rdt : dx :: mdt : mdx \div r = ef$. Also B$c$: ci :: Ah : gh, or $rdt : dx :: ndt : ndx \div r = gh$.

Now the decrement of AB is obviously $ef + gh$ — Bc, therefore we have, $dy = \frac{m}{r}\,dx + \frac{n}{r}\,dx - rdt = \frac{m+n}{r}\,dx - rdt$. Putting $(m + n) \div r = q$ and integrating we have

$$y = p + qx - rt \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

p being the constant introduced by integration.

Because $bA = a' + nt - x$, we have, by similar triangles, $v : a' + nt - x :: rdt : mdt + dx$. Substituting for y from (1) and reducing we get

$$\frac{dt}{dx} = \frac{p + qx - rt}{(a'r - mp) - (r + mq)x + r(m + n)t} = \frac{p + qx - rt}{p' - q'x + r't} \ldots\ldots (2).$$

When $y = 0$, in (1), $x = a + a'$, and $rt = ra \div n$.

$$\therefore\ p = \frac{ra}{n} - \frac{m+n}{r}.(a + a') = \frac{ra}{n} - \frac{m+n}{r}.a - \frac{m+n}{r}.\frac{m}{n}.a$$

$$= \frac{r^2a}{rn} - \frac{mn + n^2}{rn}.a - \frac{m^2 + mn}{rn}.a = \frac{a[r^2 - (m + n)^2]}{rn}.$$

If it were required to determine the length, z, of the curve, because $dz^2 = Bf^2 = Bi^2 + if^2 = Bc^2 - ci^2 + (ci - cf)^2 = r^2dt^2 - dx^2 + dx^2 - 2mdxdt + m^2dt^2 = r^2dt^2 + m^2dt^2 - 2mdxdt$,

$$\therefore\ dz = \sqrt{(r^2 + m^2)dt^2 - 2mdxdt} \ldots\ldots\ldots (3).$$

Because (2) is reducible to a homogeneous equation it may be integrated in finite terms, and hence, by substitution in (3) we have dz in functions of a single variable, which being integrated will give the length of the curve.

Messrs. S. W. Salmon, of Mt. Olive, N. J., and R. M. DeFrance, of Mercer, Pa., each sent an elegant solution of Mr. Stille's question, but as either of the above methods applies equally well to that case we reluctantly omit their publication.—Ed.]

PROBLEMS.

41. By J. P. Child, Salem, Iowa.—Given $x^2 + y = 7 \ldots\ldots\ldots (1)$,

$$x + y^2 = 11 \ldots\ldots\ldots\ldots\ldots (2),$$

to find the values of x and y.